Exercise 21

Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

$$y'' - 2y' + y = e^{2x}$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2y_c' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \to \quad y_c' = re^{rx} \quad \to \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - 2(re^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 1 = 0$$

Solve for r.

$$(r-1)^2 = 0$$

$$r = \{1\}$$

Two solutions to the ODE are e^x and xe^x. By the principle of superposition, then,

$$y_c(x) = C_1 e^x + C_2 x e^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' - 2y_p' + y_p = e^{2x} \tag{2}$$

Part (a)

Since the inhomogeneous term is an exponential function, the particular solution is $y_p = Ae^{2x}$.

$$y_p = Ae^{2x} \quad \to \quad y_p' = 2Ae^{2x} \quad \to \quad y_p'' = 4Ae^{2x}$$

Substitute these formulas into equation (2).

$$(4Ae^{2x}) - 2(2Ae^{2x}) + (Ae^{2x}) = e^{2x}$$

$$Ae^{2x} = e^{2x}$$

Match the coefficients on both sides to get an equation for A.

$$A = 1$$

The particular solution is

$$y_p = e^{2x}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 e^x + C_2 x e^x + e^{2x}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Part (b)

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x)e^x + C_2(x)xe^x$$

Differentiate it with respect to x.

$$y_p' = C_1'(x)e^x + C_2'(x)xe^x + C_1(x)e^x + C_2(x)(1+x)e^x$$

If we set

$$C_1'(x)e^x + C_2'(x)xe^x = 0, \tag{3}$$

then

$$y_p' = C_1(x)e^x + C_2(x)(1+x)e^x.$$

Differentiate it with respect to x once more.

$$y_p'' = C_1'(x)e^x + C_2'(x)(1+x)e^x + C_1(x)e^x + C_2(x)(2+x)e^x$$

Substitute these formulas into equation (2).

$$\left[C_1'(x)e^x + C_2'(x)(1+x)e^x + \cancel{C_1(x)e^x + C_2(x)(2+x)e^x}\right] - 2\left[\cancel{C_1(x)e^x + C_2(x)(1+x)e^x}\right] + \left[\cancel{C_1(x)e^x + C_2(x)xe^x}\right] = e^{2x}$$

Simplify the result.

$$C_1'(x)e^x + C_2'(x)(1+x)e^x = e^{2x} \tag{4}$$

Subtract the respective sides of equations (3) and (4) to eliminate $C_1'(x)$.

$$C_2'(x)e^x = e^{2x}$$

Solve for $C_2'(x)$.

$$C_2'(x) = e^x$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$C_2(x) = e^x$$

Solve equation (3) for $C_1'(x)$.

$$\begin{aligned} C_1'(x) &= -C_2'(x)x \\ &= -(e^x)x \\ &= -xe^x \end{aligned}$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$C_1(x) = -e^x(x-1)$$

Therefore,

$$\begin{aligned} y_p &= C_1(x)e^x + C_2(x)xe^x \\ &= \left[-e^x(x-1)\right]e^x + (e^x)xe^x \\ &= -e^{2x}(x-1) + xe^{2x} \\ &= e^{2x}, \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1e^x + C_2xe^x + e^{2x}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.